Algebraic Properties of $\ln(x)$

We can derive algebraic properties of our new function $f(x) = \ln(x)$ by comparing derivatives. We can in turn use these algebraic rules to simplify the natural logarithm of products and quotients. If a and b are positive numbers and r is a rational number, we have the following properties:

- (i) $\ln 1 = 0$ This follows from our previous discussion on the graph of $y = \ln(x)$.
- (ii) $\ln(ab) = \ln a + \ln b$
- Proof (ii) We show that $\ln(ax) = \ln a + \ln x$ for a constant $a > 0$ and any value of $x > 0$. The rule follows with $x = b$.
- Let $f(x) = \ln x, \quad x > 0$ and $g(x) = \ln(ax), \quad x > 0$. We have $f'(x) = \frac{1}{x}$ and $g'(x) = \frac{1}{ax} \cdot a = \frac{1}{x}$.
- Since both functions have equal derivatives, $f(x) + C = g(x)$ for some constant C. Substituting $x = 1$ in this equation, we get $\ln 1 + C = \ln a$, giving us $C = \ln a$ and $\ln ax = \ln a + \ln x$.

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(iii) $\ln(\frac{a}{b}) = \ln a - \ln b$

- Note that $0 = \ln 1 = \ln \frac{a}{a} = \ln(a \cdot \frac{1}{a}) = \ln a + \ln \frac{1}{a}$, giving us that $\ln \frac{1}{a} = -\ln a$.
- Thus we get $\ln \frac{a}{b} = \ln a + \ln \frac{1}{b} = \ln a - \ln b$.
- (iv) $\ln a^r = r \ln a$.
- Comparing derivatives, we see that

$$\frac{d(\ln x^r)}{dx} = \frac{rx^{r-1}}{x^r} = \frac{r}{x} = \frac{d(r \ln x)}{dx}.$$

  Hence $\ln x^r = r \ln x + C$ for any $x > 0$ and any rational number r.
- Letting $x = 1$ we get $C = 0$ and the result holds.

Example 1

Expand

$$\ln \frac{x^2\sqrt{x^2+1}}{x^3}$$

using the rules of logarithms.

- We have 4 rules at our disposal: (i) $\ln 1 = 0$, (*ii*) $\ln(ab) = \ln a + \ln b$, (iii) $\ln(\frac{a}{b}) = \ln a - \ln b$, (iv) $\ln a^r = r \ln a$.
- $\ln \frac{x^2\sqrt{x^2+1}}{x^3} \overset{\text{(iii)}}{=} \ln\left(x^2\sqrt{x^2+1}\right) - \ln\left(x^3\right)$
- $\overset{\text{(ii)}}{=} \ln(x^2) + \ln((x^2+1)^{1/2}) - \ln(x^3)$
- $\overset{\text{(iv)}}{=} 2\ln(x) + \frac{1}{2}\ln(x^2+1) - 3\ln(x)$
- $= \frac{1}{2}\ln(x^2+1) - \ln(x)$

Example 2

Express as a single logarithm:

$$\ln x + 3\ln(x+1) - \frac{1}{2}\ln(x+1).$$

- We can use our four rules in reverse to write this as a single logarithm: (i) $\ln 1 = 0$, (*ii*) $\ln(ab) = \ln a + \ln b$, (iii) $\ln(\frac{a}{b}) = \ln a - \ln b$, (iv), $\ln a^r = r\ln a$.
- $\ln x + 3\ln(x+1) - \frac{1}{2}\ln(x+1) \overset{\text{(iv)}}{=} \ln x + \ln(x+1)^3 - \ln\sqrt{x+1}$
- $\overset{\text{(ii)}}{=} \ln(x(x+1)^3) - \ln\sqrt{x+1}$
- $\overset{\text{(iii)}}{=} \ln\frac{x(x+1)^3}{\sqrt{x+1}}$

Example 3

Evaluate $\int_1^{e^2} \frac{1}{t} dt$

- From the definition of $\ln(x)$, we have

$$\int_1^{e^2} \frac{1}{t} dt = \ln(t)\Big|_1^{e^2} = \ln(e^2)$$

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$$\stackrel{(iv)}{=} 2 \ln e = 2.$$